Example

$L = \{a^n ba^n \mid n > 0\}$

Closure Properties

A set is closed over an operation if

$L_1, L_2 \in \text{class} \quad L_1 \text{ op } L_2 = L_3 \\
\Rightarrow L_3 \in \text{class}$

Example

$L = \{x \mid x \text{ is a positive even integer}\}$

$L$ is closed under

- addition?
- multiplication?
- subtraction?
- division?

Closure of Regular Languages

Theorem 4.1 If $L_1$ and $L_2$ are regular languages, then

$L_1 \cup L_2$
$L_1 \cap L_2$
$L_1L_2$
$L_1^*$

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages

$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.

$L_1 = L(r_1)$ and $L_2 = L(r_2)$

$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$

$\Rightarrow$ closed under union

$r_1 \cdot r_2$ is r.e. denoting $L_1 L_2$

$\Rightarrow$ closed under concatenation

$r_1^*$ is r.e. denoting $L_1^*$

$\Rightarrow$ closed under star-closure

complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = \delta'$
Regular languages are closed under

- reversal $L^R$
- difference $L_1 - L_2$
- right quotient $L_1 / L_2$
- homomorphism $h(L)$

Right quotient

Def: $L_1 / L_2 = \{ x | xy \in L_1 \text{ for some } y \in L_2 \}$

Example:

$$L_1 = \{ a^* b^* \cup b^* a^* \}$$  
$$L_2 = \{ b^n | n \text{ is even, } n > 0 \}$$  
$$L_1 / L_2 = \text{...}$$

**Theorem** If $L_1$ and $L_2$ are regular, then $L_1 / L_2$ is regular.

**Proof** (sketch)

$\exists$ DFA $M = (Q, \Sigma, \delta, q_0, F)$ s.t. $L_1 = L(M)$.

Construct DFA $M' = (Q, \Sigma, \delta, q'_0, F')$

For each state $i$ do

- Make $i$ the start state (representing $L_i$)

QED.
**Homomorphism**

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

**Example:**

$\Sigma = \{a, b, c\}, \Gamma = \{0, 1\}$

- $h(a) = 11$
- $h(b) = 00$
- $h(c) = 0$

$h(bc) = \quad$

$h(ab^*) = \quad$

**Questions about regular languages:**

$L$ is a regular language.

- Given $L, \Sigma, w \in \Sigma^*$, is $w \in L$?

- Is $L$ empty?

- Is $L$ infinite?

- Does $L_1 = L_2$?
Ch. 4.3 - **Identifying Nonregular Languages**

If a language $L$ is finite, is $L$ regular?

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} = \null$
- $L_2 = \{a^n b^n | n > 0\}$

**Prove that** $L_2 = \{a^n b^n | n > 0\}$ **is ?**

- Proof: Suppose $L_2$ is regular.
  $\Rightarrow \exists$ DFA $M$ that recognizes $L_2$
**Pumping Lemma:** Let \( L \) be an infinite regular language. \( \exists \) a constant \( m > 0 \) such that any \( w \in L \) with \(|w| \geq m\) can be decomposed into three parts as \( w = xyz \) with

\[
\begin{align*}
|xy| &\leq m \\
|y| &\geq 1 \\
xy^iz &\in L \quad \text{for all } i \geq 0
\end{align*}
\]

**Meaning:** Every long string in \( L \) (the constant \( m \) above corresponds to the finite number of states in \( M \) in the previous proof) can be partitioned into three parts such that the middle part can be “pumped” resulting in strings that must be in \( L \).

**To Use the Pumping Lemma to prove \( L \) is not regular:**

- Proof by Contradiction.
  
  Assume \( L \) is regular.
  
  \( \Rightarrow \) \( L \) satisfies the pumping lemma.
  
  Choose a long string \( w \) in \( L \), \(|w| \geq m\). (The choice of the string is crucial. Must pick a string that will yield a contradiction).
  
  Show that there is NO division of \( w \) into \( xyz \) (must consider all possible divisions) such that \(|xy| \leq m\), \(|y| \geq 1\) and \(xy^iz \in L \ \forall \ i \geq 0\).
  
  The pumping lemma does not hold. Contradiction!
  
  \( \Rightarrow \) \( L \) is not regular. QED.

**Example** \( L = \{a^nbc^n \mid n > 0\} \)

\( L \) is not regular.

- **Proof:**
  
  Assume \( L \) is regular.
  
  \( \Rightarrow \) the pumping lemma holds.
  
  Choose \( w = \) where \( m \) is the constant in the pumping lemma. (Note that \( w \) must be chosen such that \(|w| \geq m\).
  
  The only way to partition \( w \) into three parts, \( w = xyz \), is such that \( x \) contains 0 or more \( a \)'s, \( y \) contains 1 or more \( a \)'s, and \( z \) contains 0 or more \( a \)'s concatenated with \( cb^m \). This is because of the restrictions \(|xy| \leq m\) and \(|y| > 0\). So the partition is:

  It should be true that \( xy^iz \in L \) for all \( i \geq 0\).
Example \( L=\{a^n b^n s c^s | n, s > 0\} \)

\( L \) is not regular.

- **Proof:**
  
  Assume \( L \) is regular.
  
  \( \Rightarrow \) the pumping lemma holds.
  
  Choose \( w = \)
  
  The only way to partition \( w \) into three parts, \( w = xyz \), is such that \( x \) contains 0 or more \( a \)'s, \( y \) contains 1 or more \( a \)'s, and \( z \) contains 0 or more \( a \)'s concatenated with the rest of the string \( b^m s c^s \).

  This is because of the restrictions \( |xy| \leq m \) and \( |y| > 0 \). So the partition is:

Example \( \Sigma = \{a, b\} \), \( L=\{w \in \Sigma^* | n_a(w) > n_b(w)\} \)

\( L \) is not regular.

- **Proof:**
  
  Assume \( L \) is regular.
  
  \( \Rightarrow \) the pumping lemma holds.
  
  Choose \( w = \)
  
  So the partition is:
Example \( L = \{a^3b^n c^{n-3} | n > 3 \} \)

\( L \) is not regular.

- **Proof:**
  Assume \( L \) is regular. \( \Rightarrow \) the pumping lemma holds.

Choose \( w = a^3b^m c^{m-3} \) where \( m \) is the constant in the pumping lemma. There are three ways to partition \( w \) into three parts, \( w = xyz \). 1) \( y \) contains only \( a \)'s 2) \( y \) contains only \( b \)'s and 3) \( y \) contains \( a \)'s and \( b \)'s

We must show that each of these possible partitions lead to a contradiction. (Then, there would be no way to divide \( w \) into three parts s.t. the pumping lemma constraints were true).

**Case 1:** (\( y \) contains only \( a \)'s). Then \( x \) contains 0 to 2 \( a \)'s, \( y \) contains 1 to 3 \( a \)'s, and \( z \) contains 0 to 2 \( a \)'s concatenated with the rest of the string \( b^m c^{m-3} \), such that there are exactly 3 \( a \)'s. So the partition is:

\[
x = a^k \quad y = a^j \quad z = a^{3-k-j}b^m c^{m-3}
\]

where \( k \geq 0, j > 0, \) and \( k + j \leq 3 \) for some constants \( k \) and \( j \).

It should be true that \( xy^2z \in L \) for all \( i \geq 0 \).

\[
xy^2z = (x)(y)(y)(z) = (a^k)(a^j)(a^j)(a^{3-j-k}b^m c^{m-3}) = a^{3+j-k}b^m c^{m-3} \not\in L \text{ since } j > 0, \text{ there are too many } a \text{'s. Contradiction!}
\]

**Case 2:** (\( y \) contains only \( b \)'s) Then \( x \) contains 3 \( a \)'s followed by 0 or more \( b \)'s, \( y \) contains 1 to \( m-3 \) \( b \)'s, and \( z \) contains 3 to \( m-3 \) \( b \)'s concatenated with the rest of the string \( c^{m-3} \). So the partition is:

\[
x = a^3b^k \quad y = b^j \quad z = b^{m-k-j}c^{m-3}
\]

where \( k \geq 0, j > 0, \) and \( k + j \leq m-3 \) for some constants \( k \) and \( j \).

It should be true that \( xy^2z \in L \) for all \( i \geq 0 \).

\[
xy^2z = a^3b^j a^{m-3-j}b^m c^{m-3} \not\in L \text{ since } j > 0, \text{ there are too few } b \text{'s. Contradiction!}
\]

**Case 3:** (\( y \) contains \( a \)'s and \( b \)'s) Then \( x \) contains 0 to 2 \( a \)'s, \( y \) contains 1 to 3 \( a \)'s, and 1 to \( m-3 \) \( b \)'s, \( z \) contains 3 to \( m-1 \) \( b \)'s concatenated with the rest of the string \( c^{m-3} \). So the partition is:

\[
x = a^{3-k} \quad y = a^k b^j \quad z = b^{m-j}c^{m-3}
\]

where \( 3 \geq k > 0, \) and \( m-3 \geq j > 0 \) for some constants \( k \) and \( j \).

It should be true that \( xy^2z \in L \) for all \( i \geq 0 \).

\[
xy^2z = a^3b^j a^k b^{m-3-j}b^m c^{m-3} \not\in L \text{ since } j, k > 0, \text{ there are } b \text{'s before } a \text{'s. Contradiction!}
\]

\( \Rightarrow \) There is no partition of \( w \).

\( \Rightarrow \) \( L \) is not regular!. QED.
To Use Closure Properties to prove $L$ is not regular:

Using closure properties of regular languages, construct a language that should be regular, but for which you have already shown is not regular. Contradiction!

**Proof Outline:**
- Assume $L$ is regular.
- Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  - closure properties $\Rightarrow L'$ is regular.
  - Contradiction!
- $L$ is not regular. QED.

**Example** $L = \{a^{3^n}b^n c^{n-3} | n > 3\}$

$L$ is not regular.

**Proof:** (proof by contradiction)
- Assume $L$ is regular.
- Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$
  - $h(a) = a$  
  - $h(b) = a$  
  - $h(c) = b$
- $h(L) =$
Example \( L = \{a^m b^m a^m | m \geq 0, n \geq 0\} \)

\( L \) is not regular.

- Proof: (proof by contradiction)
  Assume \( L \) is regular.

Example: \( L_1 = \{a^n b^n a^n | n > 0\} \)

\( L_1 \) is not regular.

- Proof:
  Assume \( L_1 \) is regular.

  Goal is to try to construct \( \{a^n b^n | n > 0\} \) which we know is not regular.

  Let \( L_2 = \{a^*\} \). \( L_2 \) is regular.

  By closure under right quotient, \( L_3 = L_1 \cap L_2 = \{a^n b^n a^p | 0 \leq p \leq n, n > 0\} \) is regular.

  By closure under intersection, \( L_4 = L_3 \cap \{a^* b^*\} = \{a^n b^n | n > 0\} \) is regular.

  Contradiction, already proved \( L_4 \) is not regular!

  Thus, \( L_1 \) is not regular. QED.