Back-Propagation and Networks for Recognition

COMPSCI 527 — Computer Vision

Outline

The Soft-Max Function

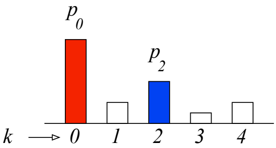
- A neural network for a K-class classifier outputs vector $\mathbf{z} = h(\mathbf{x})$ of K real numbers called *logits*, and then $\hat{y} = \arg\max_k z_k$
- During training, the output $\mathbf{z}$ is mapped to a vector $\mathbf{p}$ of *scores*, which makes formulating a good loss function easier

$$\textit{Soft-max}\ \text{function:} \qquad p_k(\mathbf{p}) = \frac{e^{z_k}}{\sum_{j=1}^{K} e^{z_j}}$$

- $p_k(\mathbf{z}) > 0$ and $\sum_{k=1}^{K} p_k(\mathbf{z}) = 1$ for all $\mathbf{z}$
- If $z_i \gg z_j$ for $j \neq i$ then $\sum_{j=1}^{K} e^{z_j} \approx e^{z_i}$
- So $p_i \approx 1$ and $p_j \approx 0$ for $j \neq i$: "Brings out the biggest:" *soft-max*
- $\arg\max_k p_k = \arg\max_k z_k$ because the soft-max is monotonic
- So the soft-max can be removed after training

The 0-1 Loss is Useless for Training

- Example: $K = 5$ classes, scores $\mathbf{p} = h(\mathbf{x}_n; \mathbf{w})$ as in figure
- True label $y_n = 2$, predicted label $\hat{y}_n = 0$ because $p_0 > p_{y_n} = p_2$. Therefore, the 0-1 loss is 1



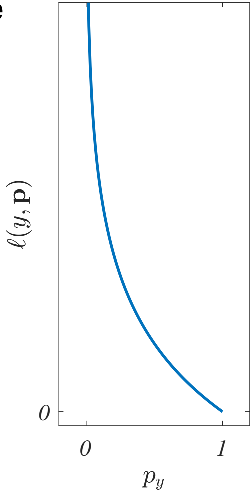
- Changing **w** by an inifinitesimal amount may *reduce* but not close the gap between p_0 and p_2: loss stays 1
- That is, $\nabla \ell_n(\mathbf{w}) = \frac{\partial \ell_{0\text{-}1}}{\partial \mathbf{w}} = 0$
- Gradient provides no information towards reducing the gap!

(Can still use 0-1 loss for *validation* or *evaluation*)

The Cross-Entropy Loss

- We compute the loss on the score vector **p**, not on the prediction $\hat{y}_n$
- Use *cross-entropy loss* on the score **p** as a proxy loss

  $\ell(y, \mathbf{p}) = -\log p_y$
- Unbounded loss for total misclassification
- Differentiable, nonzero derivative everywhere
- Meshes well with the soft-max (the layer that produces **p**)

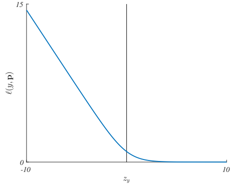


Example, Continued

- Last layer before soft-max has activations $\mathbf{z} \in \mathbb{R}^K$
- Soft-max has output $\mathbf{p} = \sigma(\mathbf{z})$ with $p_k = \frac{e^{z_k}}{\sum_{j=0}^{4} e^{z_j}} \in \mathbb{R}^5$
- $p_k > 0$ for all k and $\sum_{k=0}^{4} p_k = 1$
- Ideally, if the correct class is $y = 2$, we would like output $\mathbf{p}$ to equal $\mathbf{q} = [0, 0, 1, 0, 0]$, the *one-hot encoding* of y
- That is, $q_y = q_2 = 1$ and all other q_j are zero
- $\ell(y, \mathbf{p}) = -\log p_y = -\log p_2$
- When $\mathbf{p}$ approaches $\mathbf{q}$ we have $p_y \to 1$ and $\ell(y, \mathbf{p}) \to 0$
- When $\mathbf{p}$ is far from $\mathbf{q}$ we have $p_y \to 0$ and $\ell(y, \mathbf{p}) \to \infty$

Example, Continued

- *Cross-entropy loss meshes well with soft-max*



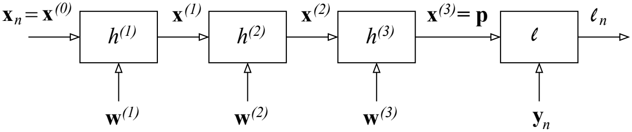
- $\ell(y, \mathbf{p}) = -\log p_y = -\log \frac{e^{z_y}}{\sum_{j=0}^{4} e^{z_j}} = \log(\sum_{j=0}^{4} e^{z_j}) - z_y$
- When $z_y \gg z_{y'}$ for all $y' \neq y$ we have $\log(\sum_{j=0}^{4} e^{z_j}) \approx \log e^{z_y} = z_y$ so that $\ell(y, \mathbf{p}) \to 0$
- When $z_y \ll z_{y'}$ for some $y' \neq y$ we have $\log(\sum_{j=0}^{4} e^{z_j}) \approx c$ (c effectively independent of z_y) so that $\ell(y, \mathbf{p}) \to c - z_y \to \infty$ linearly as $z_y \to -\infty$ (Actual plot depends on all values in $\mathbf{z}$)
- This is a "soft hinge loss" in $\mathbf{z}$ (not in $\mathbf{p}$)

Computing the Gradient of a Loss Term

- Empirical risk: $L_T(\mathbf{w}) = \frac{1}{N}\sum_{n=1}^{N} \ell(y_n, h(\mathbf{x}_n; \mathbf{w}))$
- Training summary:
  - Compute $\hat{\mathbf{w}} = \arg\min_{\mathbf{w}\in\mathbb{R}^m} L_T(\mathbf{w})$ by moving along $-\nabla L_T(\mathbf{w})$, estimated as minibatch risks $\nabla L_{B_j}(\mathbf{w})$
  - Use V to decide when to stop training (early termination)
- Regardless of what you average on, you need to compute loss gradients $\nabla \ell_n(\mathbf{w}) = \nabla \ell(y_n, h(\mathbf{x}_n; \mathbf{w}))$ and then average them (gradient of average $=$ average of gradients)
- Gradients computed by *back-propagation*, which is just the chain rule for differentiation
- The neural network $+$ loss function is the chain

Back-Propagation

$$\nabla \ell_n(\mathbf{w}) = \frac{\partial \ell_n}{\partial \mathbf{w}} = \left(\frac{\partial \ell_n}{\partial \mathbf{w}_1}, \ldots, \frac{\partial \ell_n}{\partial \mathbf{w}_J} \right)^T$$ for a network with J layers



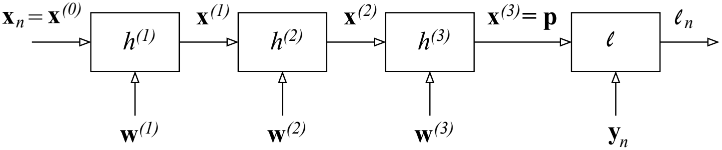
- Computations from $\mathbf{x}_n$ to ℓ_n form a chain: use the chain rule!
- Derivatives of ℓ_n w.r.t. layer j or before go through $\mathbf{x}^{(j)}$

$$\frac{\partial \ell_n}{\partial \mathbf{w}^{(j)}} = \frac{\partial \ell_n}{\partial \mathbf{x}^{(j)}} \frac{\partial \mathbf{x}^{(j)}}{\partial \mathbf{w}^{(j)}}$$

$$\frac{\partial \ell_n}{\partial \mathbf{x}^{(j-1)}} = \frac{\partial \ell_n}{\partial \mathbf{x}^{(j)}} \frac{\partial \mathbf{x}^{(j)}}{\partial \mathbf{x}^{(j-1)}} \qquad \text{(recursion!)}$$

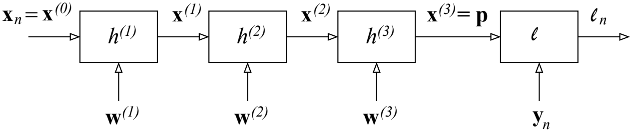
- Start: $\frac{\partial \ell_n}{\partial \mathbf{x}^{(J)}} = \frac{\partial \ell}{\partial \mathbf{p}}$

Local Jacobians



- Local computations at layer j: $\frac{\partial \mathbf{x}^{(j)}}{\partial \mathbf{w}^{(j)}}$ and $\frac{\partial \mathbf{x}^{(j)}}{\partial \mathbf{x}^{(j-1)}}$
- Partial derivatives of $h^{(j)}$ with respect to layer weights and input to the layer
- Local Jacobian matrices, can compute by knowing what the layer does
- The start of the process can be computed from knowing the loss function, $\frac{\partial \ell_n}{\partial \mathbf{x}^{(J)}} = \frac{\partial \ell}{\partial \mathbf{p}}$
- Another local Jacobian
- The rest is going recursively from output to input, one layer at a time, accumulating $\frac{\partial \ell_n}{\partial \mathbf{w}^{(j)}}$ into a vector $\frac{\partial \ell_n}{\partial \mathbf{w}}$

Back-Propagation Spelled Out for $J = 3$



$$\frac{\partial \ell_n}{\partial \mathbf{x}^{(3)}} = \frac{\partial \ell}{\partial \mathbf{p}}$$

$$\frac{\partial \ell_n}{\partial \mathbf{w}^{(3)}} = \frac{\partial \ell_n}{\partial \mathbf{x}^{(3)}} \frac{\partial \mathbf{x}^{(3)}}{\partial \mathbf{w}^{(3)}}$$

$$\frac{\partial \ell_n}{\partial \mathbf{x}^{(2)}} = \frac{\partial \ell_n}{\partial \mathbf{x}^{(3)}} \frac{\partial \mathbf{x}^{(3)}}{\partial \mathbf{x}^{(2)}}$$

$$\frac{\partial \ell_n}{\partial \mathbf{w}^{(2)}} = \frac{\partial \ell_n}{\partial \mathbf{x}^{(2)}} \frac{\partial \mathbf{x}^{(2)}}{\partial \mathbf{w}^{(2)}}$$

$$\frac{\partial \ell_n}{\partial \mathbf{x}^{(1)}} = \frac{\partial \ell_n}{\partial \mathbf{x}^{(2)}} \frac{\partial \mathbf{x}^{(2)}}{\partial \mathbf{x}^{(1)}}$$

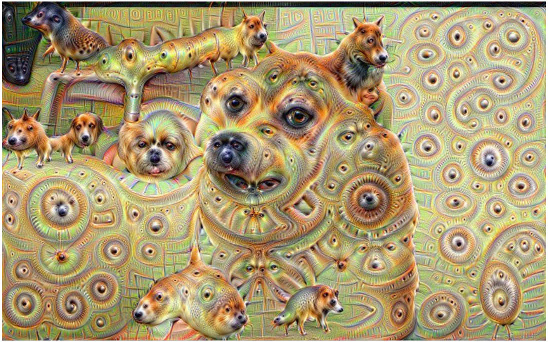
$$\frac{\partial \ell_n}{\partial \mathbf{w}^{(1)}} = \frac{\partial \ell_n}{\partial \mathbf{x}^{(1)}} \frac{\partial \mathbf{x}^{(1)}}{\partial \mathbf{w}^{(1)}}$$

$$\left(\frac{\partial \ell_n}{\partial \mathbf{x}^{(0)}} = \frac{\partial \ell_n}{\partial \mathbf{x}^{(1)}} \frac{\partial \mathbf{x}^{(1)}}{\partial \mathbf{x}^{(0)}} \right)$$

$$\frac{\partial \ell_n}{\partial \mathbf{w}} = \begin{bmatrix} \frac{\partial \ell_n}{\partial \mathbf{w}^{(1)}} \\[2ex] \frac{\partial \ell_n}{\partial \mathbf{w}^{(2)}} \\[2ex] \frac{\partial \ell_n}{\partial \mathbf{w}^{(3)}} \end{bmatrix}$$

(Jacobians in blue are local)

A Google Deep Dream Image



- Train a network to recognize animals (yields $\mathbf{w}$)
- Set $\mathbf{x}_0 =$ random noise image, $y =$ dog
- Minimize $\ell(y, h(\mathbf{x}))$ with respect to $\mathbf{x}$ rather than minimizing $L_T(\mathbf{w})$ with respect to $\mathbf{w}$

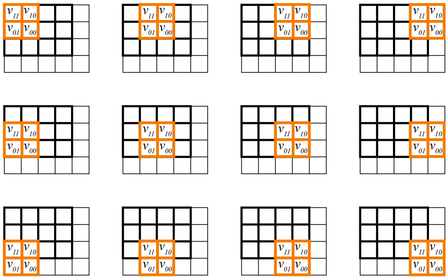
Convolutional Layers

- A fully connected layer with input $\mathbf{x} \in \mathbb{R}^d$ and output $\mathbf{y} \in \mathbb{R}^e$ has e neurons, each with d gains and one bias
- Total of $(d + 1)e$ weights to be trained in a single layer
- For images, d, e are in the order of hundreds of thousands or even millions
- Too many parameters
- *Convolutional layers* are layers restricted in a special way
- Many fewer parameters to train
- Also some justification in terms of heuristic principles (see notes)

A Convolutional Layer

- Convolution + bias: $\mathbf{a} = \mathbf{x} * \mathbf{v} + b$
- Example: 3×4 input image $\mathbf{x}$, 2×2 kernel $\mathbf{v} = \begin{bmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{bmatrix}$

["Same" style convolution]



- Do you want to see this as one convolution with $\mathbf{v}$ (plus bias) or as 12 neurons with the same weights?

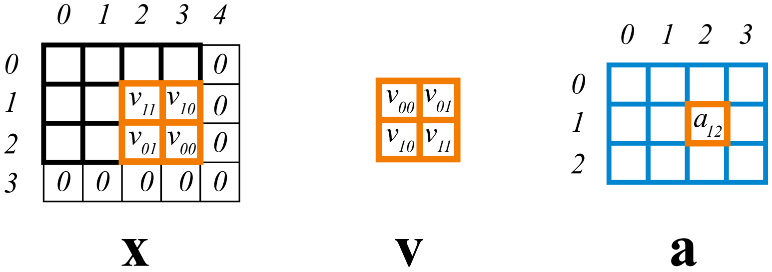
"Local" Neurons

- Neurons are now "local"
- Just means that many coefficients are zero:

0	0	0	0
0	0	v_{00}	v_{01}
0	0	v_{10}	v_{11}

- If a neuron is viewed as being connected to all input pixels, then the 12 neurons share their *nonzero* weights
- So a convolutional layer is the same as a fully-connected layer where each neuron has many weights clamped to zero, and the remaining weights are *shared* across neurons

There is Still a Gain Matrix



- Neuron number 6 (starting at 0):

  $a_{12} = v_{11}x_{12} + v_{10}x_{13} + v_{01}x_{22} + v_{00}x_{23} + b$

- Activation number six $a_{12} = V[6,:]\,\mathbf{x}$ where

$$\begin{array}{rl} \mathbf{x} & = (x_{00},\ x_{01},\ x_{02},\ x_{03},\ x_{10},\ x_{11},\ x_{12},\ x_{13},\ x_{20},\ x_{21},\ x_{22},\ x_{23})^T \\ V[6,:] & = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ v_{11},\ v_{10},\ 0,\ 0,\ v_{01},\ v_{00}) \end{array}$$

Gain Matrix for a Convolutional Layer

$$\mathbf{a} = \mathbf{x} * \mathbf{v} + b \quad \text{or} \quad \mathbf{a}_{\text{flat}} = V\mathbf{x}_{\text{flat}} + b$$

$$\begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{03} \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix}
v_{11} & v_{10} & 0 & 0 & v_{01} & v_{00} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v_{11} & v_{10} & 0 & 0 & v_{01} & v_{00} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v_{11} & v_{10} & 0 & 0 & v_{01} & v_{00} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v_{11} & v_{10} & 0 & 0 & v_{01} & v_{00} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & v_{11} & v_{10} & 0 & 0 & v_{01} & v_{00} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & v_{11} & v_{10} & 0 & 0 & v_{01} & v_{00} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & v_{11} & v_{10} & 0 & 0 & v_{01} & v_{00} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & v_{11} & v_{10} & 0 & 0 & v_{01} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_{11} & v_{10} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_{11} & v_{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_{11} & v_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_{11}
\end{bmatrix} \begin{bmatrix} x_{00} \\ x_{01} \\ x_{02} \\ x_{03} \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{20} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} + b$$

- A “regular” layer with many zeros and shared weights [Boundary neurons have fewer nonzero weights]
- Zeros cannot be changed during training
- One scalar bias instead of 12

Stride

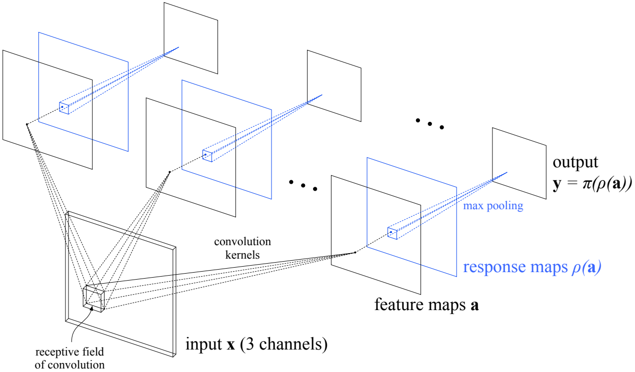
- Activation a_{ij} is often similar to $a_{i,j+1}$ and $a_{i+1,j}$
- Images often vary slowly over space
- Activations are redundant
- Reduce the redundancy by computing convolutions with a *stride* s_m greater than one
- Only compute every s_m output values in dimension m
- Output size shrinks from $d_1 \times d_2$ to about $d_1/s_1 \times d_2/s_2$
- Typically $s_m = s$ (same stride in all dimensions)
- Layers get smaller and smaller because of stride
- Multiscale image analysis, efficiency

Max Pooling

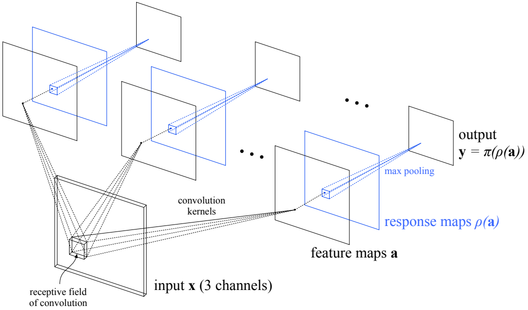
- Another way to reduce output resolution is *max pooling*
- This is a layer of its own, separate from convolution
- Consider $k \times k$ windows with stride s
- Often $s = k$ (adjacent, non-overlapping windows)
- For each window, output the maximum value
- Output is about $d_1/s \times d_2/s$
- Returns highest response in window, rather than the response in a fixed position
- More expensive than strided convolution because the entire convolution needs to be computed before the max is found in each window
- No longer very popular

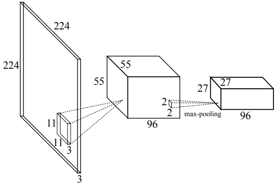
The Input Layer of AlexNet

- AlexNet *circa* 2012, classifies color images into one of 1000 categories
- Trained on ImageNet, a large database with millions of labeled images

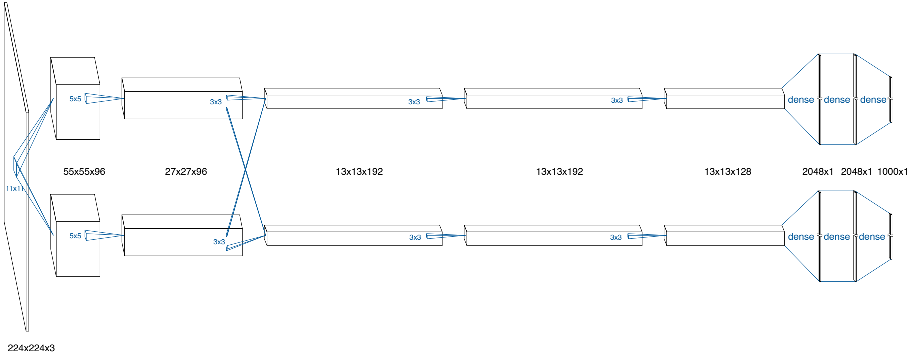


A more Compact Drawing





AlexNet

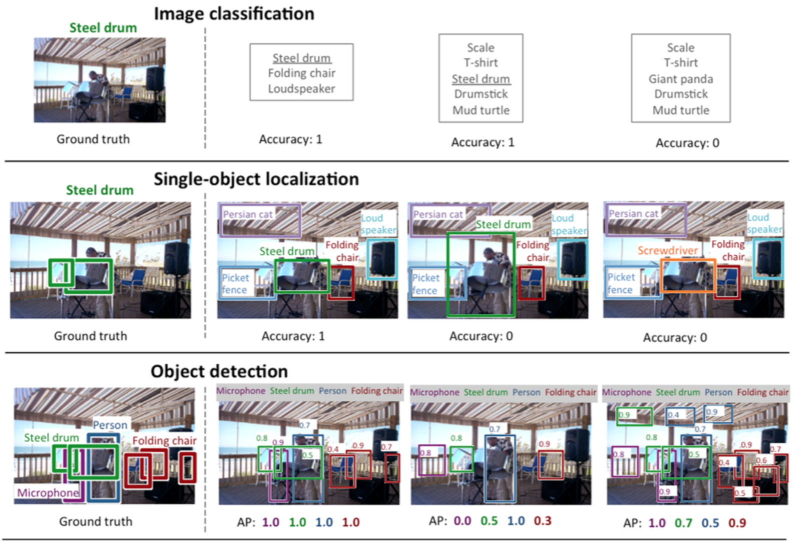

11x11
5x5
5x5
3x3
3x3
3x3
3x3
3x3
3x3
dense
dense
dense
dense
dense
dense
224x224x3
55x55x96
27x27x96
13x13x192
13x13x192
13x13x128
2048x1
2048x1
1000x1

AlexNet Numbers

- Input is $224 \times 224 \times 3$ (color image)
- First layer outputs 96 feature maps of size 55×55
- A fully-connected first layer would have about $224 \times 224 \times 3 \times 55 \times 55 \times 96 \approx 4.4 \times 10^{10}$ gains
- With convolutional kernels of size 11×11, there are only $96 \times (11^2 + 1) = 11{,}712$ weights
- That's a big deal! Locality and reuse
- Most of the complexity is in the last few, fully-connected layers, which still have millions of parameters
- More recent neural networks have much lighter final layers, but many more layers
- There are also *fully convolutional* neural networks

The State of the Art of Image Classification

- ImageNet Large Scale Visual Recognition Challenge (ILSVRC)
- Based on ImageNet,1.4 million images, 1000 categories (Fei-Fei Li, Stanford)
- Three different competitions:
  - *Classification*:
    - One label per image, 1.2M images available for training, 50k for validation, 100k withheld for testing
    - Zero-one loss for evaluation, 5 guesses allowed
  - *Localization*: Classification, plus bounding box on one instance
    Correct if $\geq$ 50% intersection/union overlap with true box
  - *Detection*: Same as localization, but find every instance in the image. Measure the fraction of mistakes (false positives, false negatives)



[Image from Russakovsky *et al.*, ImageNet Large Scale Visual Recognition Challenge, *Int'l. J. Comp. Vision* 115:211-252, 2015]

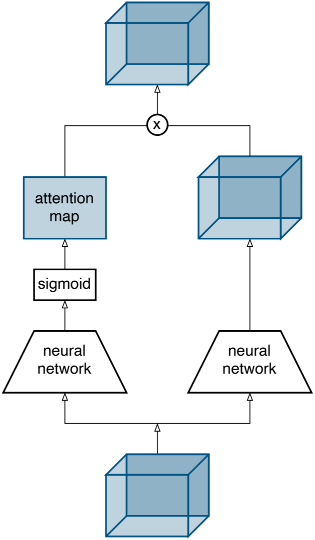
Difficulties of ILSVRC

- Images are “natural.” Arbitrary backgrounds, different sizes, viewpoints, lighting. Partially visible objects
- 1,000 categories, subtle distinctions. Example: Siberian husky and Eskimo dog
- Variations of appearance within one category can be significant (how many lamps can you think of?)
- What is the label of one image? For instance, a picture of a group of people examining a fishing rod was labeled as “reel.”

Errors on Image Classification

- Answer included in top 5:
  - 2010: 28.2 percent error rate
  - 2017: 2.3 percent (ensemble of several deep networks)
- 2021: Now we do *single top answer*, $\approx$ 10 percent error rate
- 2021 SotA has 2.4 billion weights (CoAtNet, Dai *et al.*, Google Brain). Attention mechanisms are gaining interest
  - Attention: Inputs to a layer are weighted by a learnable *attention mask* that emphasizes relevant parts of the image
  - Just add the mask, then SGD will tune it to do the right thing
- Improvement results from both architectural insights (residuals, attention, positional encoding, ...) and persistent engineering
- A book on “tricks of the trade in deep learning!”
- Problem solved? Only on ImageNet!
- “Meta-overfitting”

Attention



- The right network is the usual network (or part of it)
- The left network is similar but outputs a single channel with values in $(0, 1)$
- The product multiplies each (scalar) pixel in the attention map with each (vector) pixel in the activation map
- The rest is the same: Train the whole system by SGD