

Section: Regular Languages

Regular Expressions

Method to represent strings in a language

- + union (or)
- o concatenation (AND) (can omit)
- * star-closure (repeat 0 or more times)

Example:

$$(a + b)^* \circ a \circ (a + b)^*$$

$(a+b)^* a (a+b)^*$

strings w/ at least one a

Example:

$$(aa)^*$$

even number of a's

Definition Given Σ ,

1. $\emptyset, \lambda, a \in \Sigma$ are R.E.

2. If r and s are R.E. then

- $r+s$ is R.E.
- rs is R.E.
- (r) is a R.E.
- r^* is R.E.

3. r is a R.E. iff it can be derived from
(1) with a finite number of
applications of (2).

Definition: $L(r)$ = language denoted by R.E. r .

1. $\emptyset, \{\lambda\}, \{a\}$ are L denoted by a R.E.
2. if r and s are R.E. then
 - (a) $L(r+s) = L(r) \cup L(s)$
 - (b) $L(rs) = L(r) \circ L(s)$
 - (c) $L((r)) = L(r)$
 - (d) $L((r)^*) = (L(r))^*$

Precedence Rules

- * highest

- o

- +

Example:

$$ab^* + c = (a(b^*)) + c$$

$$\Rightarrow \text{OR } b^*(ab^* + ab^*ab^* + \dots)ab$$

Examples:

1. $\Sigma = \{a, b\}$, $\{w \in \Sigma^* \mid w \text{ has an odd number of } a\text{'s followed by an even number of } b\text{'s}\}$.

$$a(aa)^*(bb)^*$$

2. $\Sigma = \{a, b\}$, $\{w \in \Sigma^* \mid w \text{ has no more than 3 } a\text{'s and must end in } ab\}$.

$$(b^* + b^*ab^* + b^*ab^*ab^*)ab$$

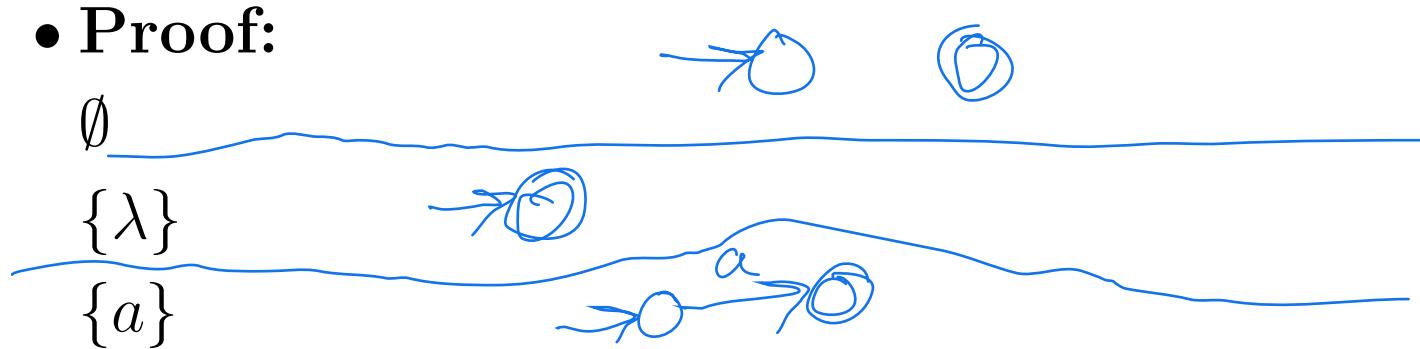
3. Regular expression for all integers (including negative)

$$0 + (- + 2)(1+2+\dots+9)(0+1+\dots+9)^*$$

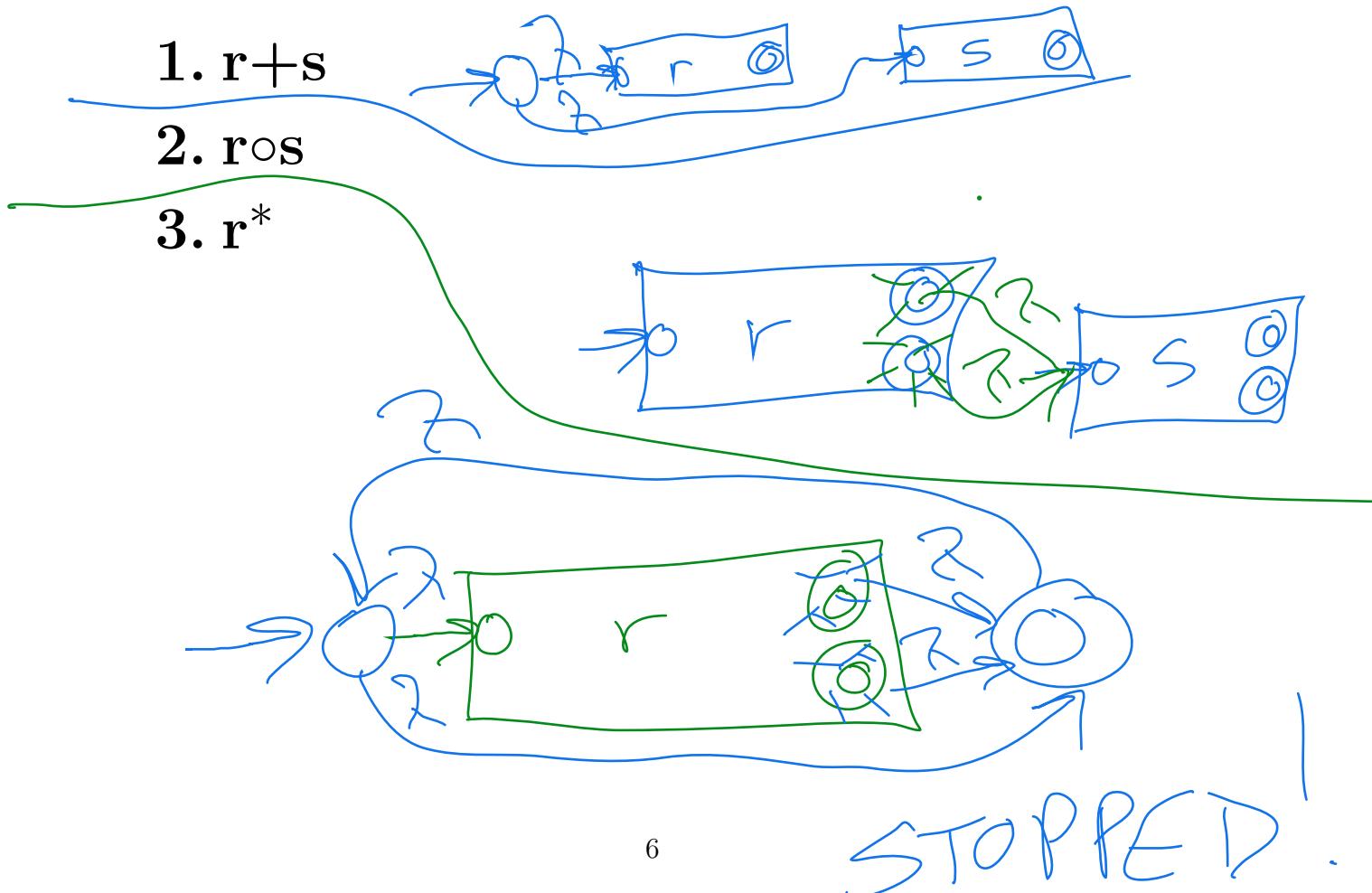
Section 3.2 Equivalence of DFA and R.E.

Theorem Let r be a R.E. Then \exists NFA M s.t. $L(M) = L(r)$.

- Proof:



Suppose r and s are R.E.



Example

$$ab^* + c$$

Theorem Let L be regular. Then \exists R.E. r s.t. $L=L(r)$.

Proof Idea: remove states sucessively until two states left

- **Proof:**

L is regular

$\Rightarrow \exists$

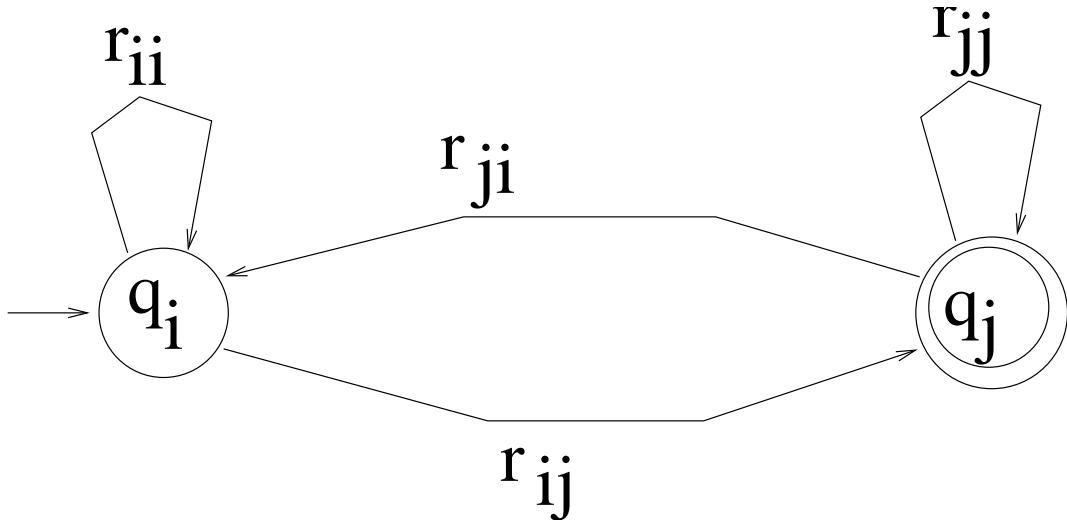
1. Assume M has one final state and $q_0 \notin F$

2. Convert to a generalized transition graph (GTG), all possible edges are present.

If no edge, label with

Let r_{ij} stand for label of the edge from q_i to q_j

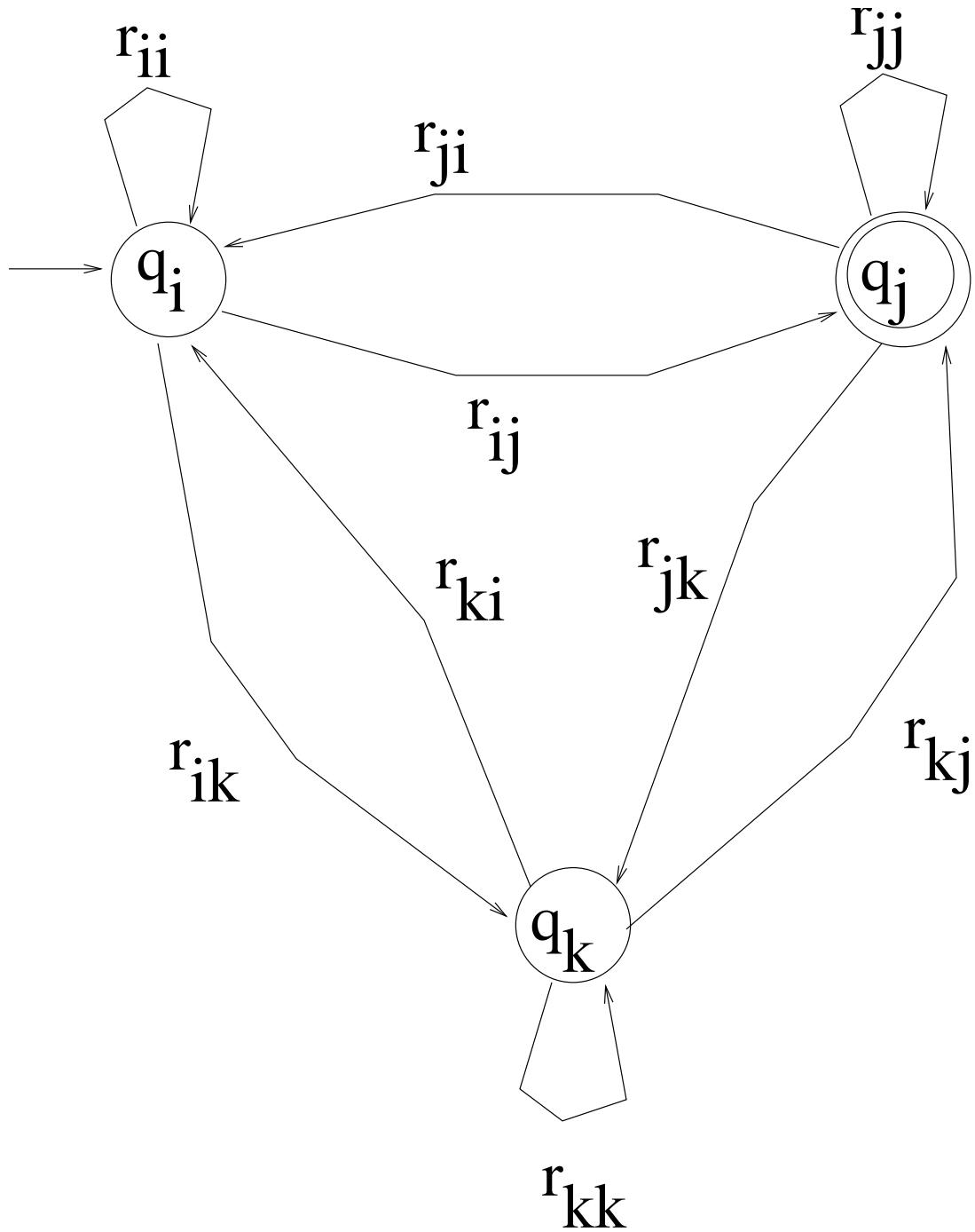
3. If the GTG has only two states, then it has the following form:



In this case the regular expression is:

$$r = (r_{ii}^* r_{ij} r_{jj}^* r_{ji})^* r_{ii}^* r_{ij} r_{jj}^*$$

4. If the GTG has three states then it must have the following form:



REPLACE

WITH

r_{ii}	$r_{ii} + r_{ik}r_{kk}^*r_{ki}$
r_{jj}	$r_{jj} + r_{jk}r_{kk}^*r_{kj}$
r_{ij}	$r_{ij} + r_{ik}r_{kk}^*r_{kj}$
r_{ji}	$r_{ji} + r_{jk}r_{kk}^*r_{ki}$

remove state q_k

5. If the GTG has four or more states, pick a state q_k to be removed (not initial or final state).

For all $o \neq k, p \neq k$ use the rule r_{op} replaced with $r_{op} + r_{ok}r_{kk}^*r_{kp}$ with different values of o and p.

When done, remove q_k and all its edges. Continue eliminating states until only two states are left. Finish with step 3.

6. In each step, simplify the regular expressions r and s with:

$$r + r = r$$

$$s + r^*s =$$

$$r + \emptyset =$$

$$r\emptyset =$$

$$\emptyset^* =$$

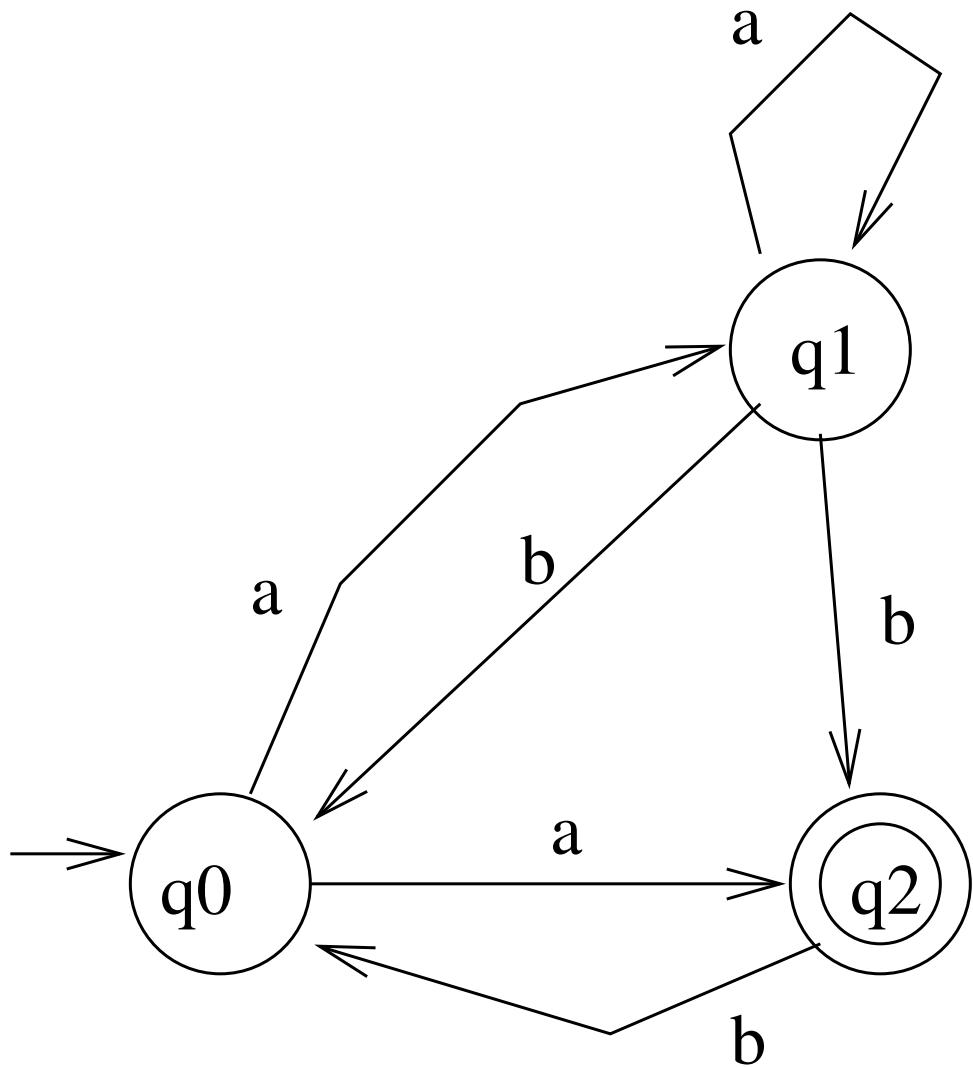
$$r\lambda =$$

$$(\lambda + r)^* =$$

$$(\lambda + r)r^* =$$

and similar rules.

Example:



Grammar $G=(V,T,S,P)$

V variables (nonterminals)

T terminals

S start symbol

P productions

Right-linear grammar:

all productions of form

$A \rightarrow xB$

$A \rightarrow x$

where $A,B \in V$, $x \in T^*$

Left-linear grammar:

all productions of form

$$A \rightarrow Bx$$

$$A \rightarrow x$$

where $A, B \in V$, $x \in T^*$

Definition:

A regular grammar is a right-linear or left-linear grammar.

Example 1:

$$G = (\{S\}, \{a, b\}, S, P), P =$$
$$S \rightarrow abS$$
$$S \rightarrow \lambda$$
$$S \rightarrow Sab$$

Example 2:

$$G = (\{S, B\}, \{a, b\}, S, P), P =$$
$$\begin{array}{c} S \rightarrow aB \mid bS \mid \lambda \\ B \rightarrow aS \mid bB \end{array}$$

Theorem: L is a regular language iff \exists regular grammar G s.t. $L=L(G)$.

Outline of proof:

- (\Leftarrow) Given a regular grammar G
 - Construct NFA M
 - Show $L(G)=L(M)$
- (\Rightarrow) Given a regular language
 - \exists DFA M s.t. $L=L(M)$
 - Construct reg. grammar G
 - Show $L(G) = L(M)$

Proof of Theorem:

(\Leftarrow) Given a regular grammar G
 $G = (V, T, S, P)$

$$V = \{V_0, V_1, \dots, V_y\}$$

$$T = \{v_0, v_1, \dots, v_z\}$$

$$S = V_0$$

Assume G is right-linear
(see book for left-linear case).

Construct NFA M s.t. $L(G) = L(M)$

If $w \in L(G)$, $w = v_1 v_2 \dots v_k$

$M = (V \cup \{V_f\}, T, \delta, V_0, \{V_f\})$

V_0 is the start (initial) state

For each production, $V_i \rightarrow aV_j$,

For each production, $V_i \rightarrow a$,

Show $L(G) = L(M)$

Thus, given R.G. G,

$L(G)$ is regular

(\Rightarrow) Given a regular language L

\exists DFA M s.t. $L=L(M)$

$M=(Q, \Sigma, \delta, q_0, F)$

$Q=\{q_0, q_1, \dots, q_n\}$

$\Sigma = \{a_1, a_2, \dots, a_m\}$

Construct R.G. G s.t. $L(G) = L(M)$

$G=(Q, \Sigma, q_0, P)$

if $\delta(q_i, a_j) = q_k$ then

if $q_k \in F$ then

Show $w \in L(M) \iff w \in L(G)$

Thus, $L(G) = L(M)$.

QED.

Example

$$G = (\{S, B\}, \{a, b\}, S, P), P =$$
$$\begin{array}{c} S \rightarrow aB \mid bS \mid \lambda \\ B \rightarrow aS \mid bB \end{array}$$

Example:

