Section: Properties of Regular Languages

Example

$L = \{a^n b a^n \mid n > 0\}$

NOT regular!

Closure Properties

A set is closed over an operation if

$\mathbf{L}_1, \mathbf{L}_2 \in$ **class**
$\mathbf{L}_1$ **op** $\mathbf{L}_2 = \mathbf{L}_3$
$\Rightarrow \mathbf{L}_3 \in$ **class**

L={x | x is a positive even integer}

L is closed under

- addition? yes
- multiplication? yes
- subtraction? No $6-10=-4$
- division? NO

Closure of Regular Languages

Theorem 4.1 If L_1 and L_2 are regular languages, then

$$L_1 \cup L_2$$
$$L_1 \cap L_2$$
$$L_1 L_2$$
$$\bar{L}_1$$
$$L_1^*$$

are regular languages.

Proof(sketch)

L_1 and L_2 are regular languages
$\Rightarrow \exists$ reg. expr. r_1 and r_2 s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1 r_2$ is r.e. denoting $L_1 L_2$
$\Rightarrow$ closed under concatenation
r_1^* is r.e. denoting L_1^*
$\Rightarrow$ closed under star-closure

complementation:

L_1 is reg. lang.

$\Rightarrow \exists$ DFA M s.t. $L_1 = L(M)$

Construct M' s.t.

final states → non-final

non-final states → final

have M' DFA for $\overline{L_1}$

show $w \in L(M') \Leftrightarrow w \in \overline{L}$

intersection:

L_1 and L_2 are reg. lang.

$\Rightarrow \exists$ DFA M_1 and M_2 s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1=(Q,\Sigma,\delta_1, q_0, F_1)$

$M_2=(P,\Sigma,\delta_2, p_0, F_2)$

Construct $M'=(Q',\Sigma,\delta', (q_0,p_0), F')$

$Q' =$ $Q \times P$

δ':

$\delta'((q_i,p_j),a) = (q_k, p_l)$ if

$\delta_1(q_i,a) = q_k \in M_1$ and

$\delta_2(p_j,a) = p_l \in M_2$

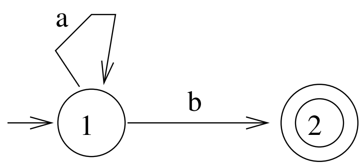
$F' = \{(q_i,p_j) \in Q' \mid q_i \in F_1 \text{ and } p_j \in F_2\}$

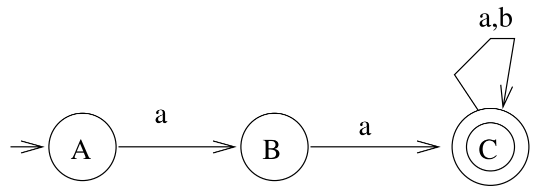
show $w \in L(M') \Leftrightarrow w \in L_1 \cap L_2$

$\Rightarrow$ closed under intersection

STOPPED

Example:





Regular languages are closed under

reversal	$\mathbf{L}^R$
difference	$\mathbf{L}_1$-$\mathbf{L}_2$
right quotient	$\mathbf{L}_1/\mathbf{L}_2$
homomorphism	$\mathbf{h(L)}$

Right quotient

Def: $\mathbf{L}_1/\mathbf{L}_2 = \{x | xy \in \mathbf{L}_1$ **for some** $y \in \mathbf{L}_2\}$

Example:

$\mathbf{L}_1 = \{a^* b^* \cup b^* a^*\}$
$\mathbf{L}_2 = \{b^n | n$ **is even,** $n > 0\}$
$\mathbf{L}_1/\mathbf{L}_2 =$

Theorem If L_1 and L_2 are regular, then L_1/L_2 is regular.

Proof (sketch)

$\exists$ DFA M=(Q,Σ,δ,q_0,F) s.t. L_1 = L(M).

Construct DFA M'=(Q,Σ,δ,q_0,F')

For each state i do

Make i the start state (representing $L_i^{'}$)

QED.

Homomorphism

Def. Let Σ, Γ be alphabets. A homomorphism is a function

$$\mathbf{h:}\Sigma \to \Gamma^*$$

Example:

$\Sigma = \{a, b, c\}$**,** $\Gamma = \{0, 1\}$

h(a)=11
h(b)=00
h(c)=0

h(bc) =

h(ab*) =

Questions about regular languages :

L is a regular language.

- **Given L, Σ, w$\in \Sigma^*$, is w$\in$L?**

- **Is L empty?**

- **Is L infinite?**

- **Does $L_1 = L_2$?**

Identifying Nonregular Languages

If a language L is finite, is L regular?

If L is infinite, is L regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} =$
- $L_2 = \{a^n b^n | n > 0\}$

Prove that $L_2 = \{a^n b^n | n > 0\}$ is ?

- **Proof: Suppose L_2 is regular.**
  $\Rightarrow \exists$ **DFA M that recognizes** L_2

Pumping Lemma: Let L be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$\begin{aligned}&|xy| \leq m\\&|y| \geq 1\\&xy^iz \in L \text{ for all } i \geq 0\end{aligned}$$

To Use the Pumping Lemma to prove L is not regular:

- **Proof by Contradiction.**

  Assume L is regular.

  $\Rightarrow$ L satisfies the pumping lemma.

  Choose a long string w in L,
  $|w| \geq m$.

  Show that there is NO division of w into xyz (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in$L $\forall$ $i \geq 0$.

  The pumping lemma does not hold. Contradiction!

  $\Rightarrow$ L is not regular. QED.

Example L=$\{a^n cb^n | n > 0\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ **the pumping lemma holds.**

  Choose $w =$

Example L=$\{a^n b^{n+s} c^s | n, s > 0\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ **the pumping lemma holds.**

  Choose w=

  So the partition is:

Example $\Sigma = \{a, b\}$,
L=$\{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose w=

  So the partition is:

Example L=$\{a^3b^nc^{n-3}|n > 3\}$

(shown in detail on handout)

L is not regular.

To Use Closure Properties to prove L is not regular:

- **Proof Outline:**

  Assume L is regular.

  Apply closure properties to L and other regular languages, constructing L' that you know is not regular.

  closure properties $\Rightarrow$ L' is regular.

  Contradiction!

  L is not regular. QED.

Example L=$\{a^3b^nc^{n-3}|n > 3\}$

L is not regular.

- **Proof: (proof by contradiction)**
  Assume L is regular.
  Define a homomorphism $h : \Sigma \to \Sigma^*$
  $h(a) = a \;\; h(b) = a \;\; h(c) = b$
  $h(L) =$

Example L=$\{a^n b^m a^m | m \geq 0, n \geq 0\}$

L is not regular.

- **Proof: (proof by contradiction)**
  Assume L is regular.

Example: $L_1 = \{a^n b^n a^n | n > 0\}$

L_1 **is not regular.**