Section: Properties of Regular Languages

Example

$L = \{a^n b a^n \mid n > 0\}$

NOT regular!

Closure Properties

A set is closed over an operation if

$$\mathbf{L}_1, \mathbf{L}_2 \in \textbf{class}$$
$$\mathbf{L}_1 \textbf{ op } \mathbf{L}_2 = \mathbf{L}_3$$
$$\Rightarrow \mathbf{L}_3 \in \textbf{class}$$

L={x | x is a positive even integer}

L is closed under

addition? yes
multiplication? yes
subtraction? No $6-10=-4$
division? NO

Closure of Regular Languages

Theorem 4.1 If L_1 and L_2 are regular languages, then

$L_1 \cup L_2$
$L_1 \cap L_2$
L_1L_2
$\bar{L}_1$
L_1^*

are regular languages.

Proof(sketch)

$\mathbf{L}_1$ **and** $\mathbf{L}_2$ **are regular languages**
$\Rightarrow \exists$ **reg. expr.** r_1 **and** r_2 **s.t.**
$\mathbf{L}_1 = \mathbf{L}(r_1)$ **and** $\mathbf{L}_2 = \mathbf{L}(r_2)$
$r_1 + r_2$ **is r.e. denoting** $\mathbf{L}_1 \cup \mathbf{L}_2$ ✓
$\Rightarrow$ **closed under union**
$r_1 r_2$ **is r.e. denoting** $\mathbf{L}_1\mathbf{L}_2$
$\Rightarrow$ **closed under concatenation** ✓
r_1^* **is r.e. denoting** $\mathbf{L}_1^*$
$\Rightarrow$ **closed under star-closure** ✓

complementation:

L_1 is reg. lang.

$\Rightarrow \exists$ DFA M s.t. $L_1 = L(M)$

Construct M' s.t.

final states $\Rightarrow$ non-final

non-final states $\Rightarrow$ final

have M' DFA for $\overline{L_1}$

show $w \in L(M') \Leftrightarrow w \in \overline{L}$

intersection:

L_1 and L_2 are reg. lang.

$\Rightarrow \exists$ DFA M_1 and M_2 s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = Q \times P$

δ':

$\delta'((q_i, p_j), a) = (q_k, p_l)$ if

$\delta_1(q_i, a) = q_k \in M_1$ and

$\delta_2(p_j, a) = p_l \in M_2$

$F' = \{(q_i, p_j) \in Q' \mid q_i \in F_1 \text{ and } p_j \in F_2\}$

show $w \in L(M') \Longleftrightarrow w \in L_1 \cap L_2$

$\Longrightarrow$ closed under intersection

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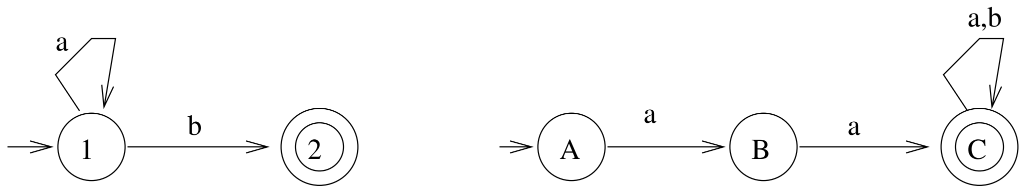
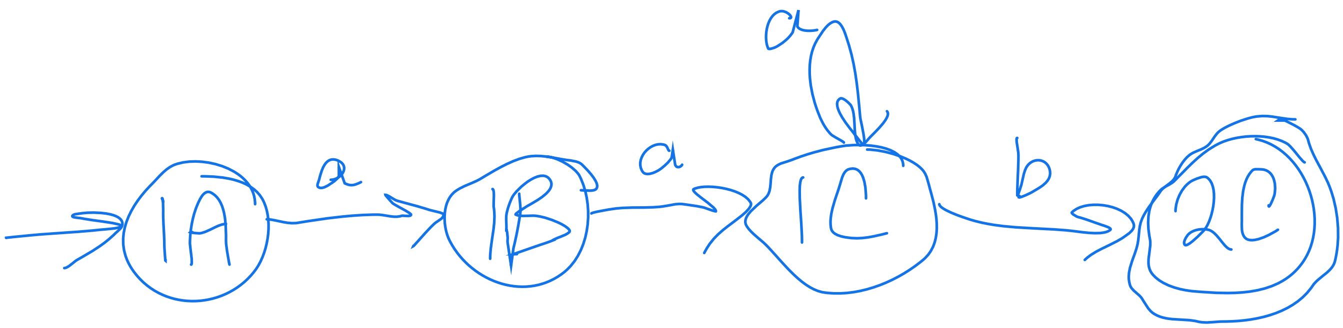
Example:





Regular languages are closed under

reversal	$\mathbf{L}^R$
difference	$\mathbf{L}_1$-$\mathbf{L}_2$
right quotient	$\mathbf{L}_1/\mathbf{L}_2$
homomorphism	$\mathbf{h(L)}$

S

Right quotient

Def: $\mathbf{L}_1/\mathbf{L}_2 = \{x | xy \in \mathbf{L}_1$ **for some** $y \in \mathbf{L}_2\}$

Example:

$\mathbf{L}_1 = \{a^*b^* \cup b^*a^*\}$
$\mathbf{L}_2 = \{b^n | n$ **is even,** $n > 0\}$
$\mathbf{L}_1/\mathbf{L}_2 = \{a^* b^*\}$

Theorem If L_1 and L_2 are regular, then L_1/L_2 is regular.

Proof (sketch)

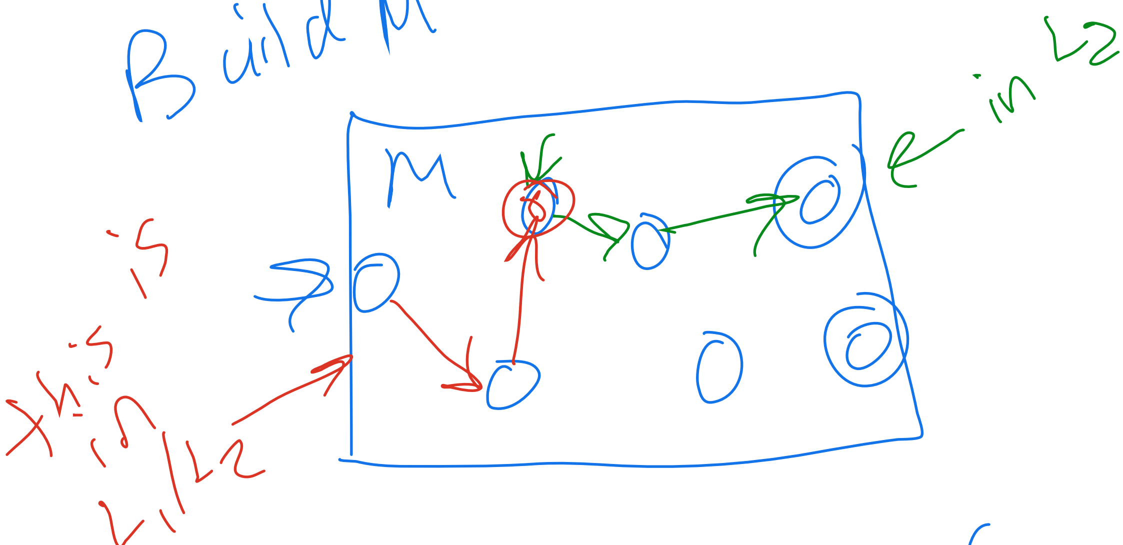
$\exists$ DFA M=(Q,Σ,δ,q_0,F) s.t. $L_1 = L(M)$.

Construct DFA M'=(Q,Σ,δ,q_0,F') for L_1/L_2

For each state i do

Make i the start state (representing L_i')

Build M'



if $L_i' \cap L_2 \neq \emptyset$ then put q_i in F' in M'

QED.

Homomorphism

Def. Let Σ, Γ be alphabets. A homomorphism is a function

$$\mathbf{h}: \Sigma \to \Gamma^*$$

Example:

$$\Sigma = \{a, b, c\}, \ \Gamma = \{0, 1\}$$

$$\mathbf{h(a)=11}$$
$$\mathbf{h(b)=00}$$
$$\mathbf{h(c)=0}$$

$$\mathbf{h(bc)} = 000$$

$$\mathbf{h(ab^*)} = 11(00)^*$$

Questions about regular languages :

L is a regular language.

- **Given L, Σ, w$\in \Sigma^*$, is w$\in$L?**

Construct a DFA for L,
test w to see if accepted

- **Is L empty?**

Construct a DFA
If there is a path from initial state
to a Final state ⇒ then not empty
use DFS or BFS

- **Is L infinite?**

Construct a DFA
if there is a cycle ⇒ then infinite
language

- **Does $L_1 = L_2$?**

Construct $L_3 = (L_1 \cap \overline{L_2}) \cup (L_2 \cap \overline{L_1})$
if $L_3 = \phi$ then $L_1 = L_2$

Identifying Nonregular Languages

If a language L is finite, is L regular?

yes

If L is infinite, is L regular?

maybe, maybe not !

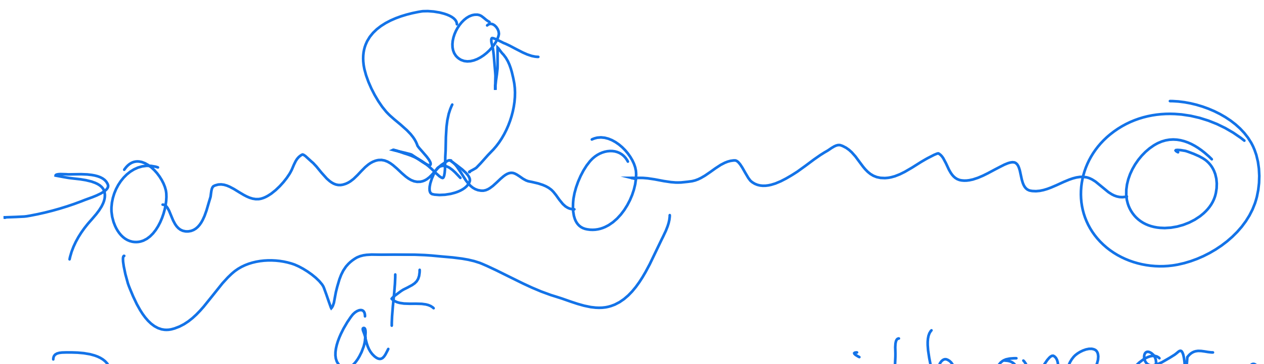
- $L_1 = \{a^n b^m | n > 0, m > 0\}$ = aa^*bb^* regular
- $L_2 = \{a^n b^n | n > 0\}$ NOT regular

Prove that $L_2 = \{a^n b^n | n > 0\}$ is ?

- **Proof: Suppose L_2 is regular.**
  $\Rightarrow \exists$ DFA M that recognizes L_2

M finite no. of states, K states

Consider a long string $a^K b^K \in L_2$



$\Rightarrow$ has to be a loop with one or more a's

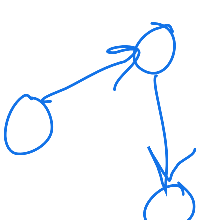
$\Rightarrow$ we can generate a string with more than K a's!

for $a^K b^K$ we get $a^{K+j} b^K \notin L_2$

Contradiction.

$\Rightarrow L_2$ is not regular

3 states



add 3rd arc, has to be a loop

no. of states in a DFA

Pumping Lemma: Let L be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

long string

$$|xy| \leq m$$
$$|y| \geq 1$$ part we pump, has to be something there
$$xy^iz \in L \textbf{ for all } i \geq 0$$

m is same thing as K from before

$$W = \overbrace{\underbrace{aaaa}_{x}\underbrace{a \ldots}_{y}\underbrace{\ldots a}_{}}^{m \text{ states in DFA}}\!\!\underbrace{\,bb \ldots\ldots b}_{z}$$

To Use the Pumping Lemma to prove L is not regular:

- **Proof by Contradiction.**

  Assume L is regular.

  $\Rightarrow$ L satisfies the pumping lemma.

  Choose a long string w in L, $|w| \geq m$.

  Show that there is NO division of w into xyz (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in$L $\forall\ i \geq 0$.

  The pumping lemma does not hold. Contradiction!

  $\Rightarrow$ L is not regular. QED.

Example L=$\{a^n c b^n | n > 0\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose $w =$ $a^m c b^m \in L$

make sure string has m in it, a m from Pumping lemma

general way to show all partitions

$x = a^k$ $\quad y = a^j$ $\quad z = a^{m-k-j} c b^m$

$j > 0$

$xy^iz \in L \;\; \forall i$

find an i that doesn't work

$i = 0 \quad xy^0z = a^{m-j} c b^m \notin L$

Contradiction! ✓

$\Rightarrow$ L is not regular

Note that $i=2$ also works to find a contradiction

you only have to find one i that doesn't work

Example L=$\{a^n b^{n+s} c^s | n, s > 0\}$

also work

or $a^m b^{m+1} c$

$a^m b^{m+3} c^3$

$a b^{m+1} c^m$ not so good proof messier

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ the pumping lemma holds.

  Choose w= $a^m b^{2m} c^m$

  So the partition is:

general for all partitions

$x = a^k \quad y = a^j \quad z = a^{m-k-j} b^{2m} c^m$

$j > 0$

should be true $xy^iz \in L \quad \forall i$

$i = 2 \quad xy^2z = a^{m+j} b^{2m} c^m \notin L$

Reason $\Rightarrow$ no of a's > no of c's

OR no of b's $\neq$ no of a's + no of c's

Contradiction!

$\Rightarrow$ L is not regular!

Example $\Sigma = \{a, b\}$,
L$=\{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

L is not regular.

- **Proof:**

  Assume L is regular.

  $\Rightarrow$ **the pumping lemma holds.**

  Choose $w = b^m a^{m+1} \in L$

  So the partition is:

OR $w = a^m b^{m+1}$ ✗
$a^{m+1} b^m$ ✓

$x = b^k \quad y = b^j \quad z = b^{m-k-j} a^{m+1}$

$j > 0$

$\forall i \quad xy^iz \in L$

$i = 2 \quad xy^2z = b^{m+j} a^{m+1} \notin L$

no of b's > no of a's

Contradiction

$\Rightarrow$ L is not regular

Note $i=0$ does not give contradiction

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Example L=$\{a^3b^nc^{n-3}|n > 3\}$

(shown in detail on handout)

L is not regular.

assume L is regular

$w = a^3 b^m c^{m-3}$ $\qquad w = xyz$

aaabb.......bcc......c

x

x $\quad$ y

Case 1 $\quad x = a^k \quad y = a^j \quad z = a^{3-k-j} b^m c^{m-3}$

$i = 2 \Rightarrow$ more than 3 a's

Case 2 $\quad x = a^3 b^k \quad y = b^j \quad z = b^{m-j-k} c^{m-3}$

etc lots of cases! find an i for each one

To Use Closure Properties to prove L is not regular:

- **Proof Outline:**

  Assume L is regular.

  Apply closure properties to L and other regular languages, constructing L' that you know is not regular.

  closure properties $\Rightarrow$ L' is regular.

  Contradiction!

  L is not regular. QED.

Example L=$\{a^3b^nc^{n-3}|n>3\}$

L is not regular.

- **Proof: (proof by contradiction)**
  Assume L is regular.
  Define a homomorphism $h:\Sigma\rightarrow\Sigma^*$
  $h(a)=a\ \ h(b)=a\ \ h(c)=b$
  $h(L)=\{a^3a^nb^{n-3}\mid n>3\}$
  $=\{a^{n+3}b^{n-3}\mid n>3\}$

h(L) is regular

$L'=h(L)\{b^6\}=\{a^{n+3}b^{n+3}\mid n>3\}$

L' is regular

$L''=\{ab, a^2b^2, \cdots a^6b^6\}$ reg

$L'\cup L''=\{a^nb^n\mid n>0\}$ is reg

Contradiction $\Rightarrow$ not regular

$\Rightarrow$ L is not regular.

Example L=$\{a^n b^m a^m | m \geq 0, n \geq 0\}$

L is not regular.

- **Proof: (proof by contradiction)**
  Assume L is regular.

$L1 = \{bb^*aa^*\}$

$L2 = L \cap L1 = \{b^m a^m \mid m > 0\}$

$h(a) = b \quad h(b) = a$

$h(L2) = \{a^n b^n \mid n > 0\}$ should be regular

contradiction!

$\Rightarrow$ L is not regular

Example: $L_1 = \{a^n b^n a^n | n > 0\}$

L_1 **is not regular.**

$L_2 = \{a^*\}$ is reg

$L_3 = L_1 \setminus L_2 = \{a^n b^n a^p \mid 0 \le p \ne n, n > 0\}$

L_3 should be regular

$L_4 = L_3 \cap \{a^* b^*\} = \{a^n b^n \mid n > 0\}$

should be reg.

Contradiction

$\Longrightarrow L_1$ is not regular